J. Appl. Maths Mechs Vol. 57, No. 1, pp. 67–74, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

THE FORCES ACTING ON AN OBSTACLE IN STRATIFIED FLOW†

A. M. TER-KRIKOROV

Moscow

(*Received* 10 *February* 1992)

The three-dimensional steady flow of a heavy incompressible ideal fluid past an obstacle with a rigid boundary with general restrictions on the density and velocity distributions of the incident flow is considered. A set of two non-linear second-order equations which describe the flow pattern is derived. Formulation of the boundary-value problem is discussed. Formulae for calculating the forces acting on the obstacle are derived. The simplifying assumptions associated with approximating the obstacle by a system of dipoles distributed over the barrier surface are investigated. As an example, the flow of an unbounded exponentially stratified fluid around a sphere is considered. Assuming the stratification parameter to be small, the main term in the asymptotic formula which expresses the dependence of the resistance on the sphere radius and the stratification parameters is calculated.

1. THE FUNDAMENTAL EQUATIONS

CONSIDER the three-dimensional stratified steady flow of a heavy incompressible ideal fluid with a free boundary and a finite number of interfaces within the fluid on which abrupt changes in density ρ and tangential velocity $\mathbf{v}$ occur. To simplify the formulations, we will assume that a fluid of zero density is at rest above the free boundary. In the case of a finite depth, the rigid bottom is horizontal. There is a flow around the source of disturbance which is located in the fluid and occupies the region Ω. We choose the origin of the Cartesian coordinates on the undisturbed free boundary; the z axis is directed vertically upwards while the x axis is directed along the undisturbed flow. By choosing a characteristic linear dimension H_0, velocity U_0 and density ρ_0, we can write the equation of fluid flow in dimensionless variables. The density ρ and velocity v_x of the undisturbed one-dimensional flow are defined by piecewise-smooth functions $\rho(z)$ and $U(y, z)$ which undergo abrupt changes when passing through the $z = z_k$ planes, and in this case, $-\infty < z_n < \ldots < z_0 = 0$, $d\rho \geqslant 0$, $U \geqslant c > 0$.

As $x \to -\infty$ the flow is asymptotically undisturbed; the x coordinate increases from $-\infty$ to $+\infty$ along the trajectory of the fluid particles. The trajectory which passes through an arbitrary point (x, y, z) as $x \to -\infty$ asymptotically approaches a straight line $Y = \eta(x, y, z)$, $Z = \zeta(x, y, z)$. Since the flow is steady, the functions $\nabla\eta$ and $\nabla\zeta$ maintain their constant values along the trajectories, being the integrals of the equations of motion, and consequently, $(\mathbf{v}, \nabla\eta) = 0$, $(\mathbf{v}, \nabla\zeta) = 0$.

Suppose that $\nabla\eta$ and $\nabla\zeta$ are independent quantities; we shall assume that

$$\mathbf{v} = U(\eta, \zeta)\mathbf{b}, \quad \mathbf{b} = [\nabla\eta, \nabla\zeta] \tag{1.1}$$

In view of (1.1), the equation of continuity is satisfied. Since any sufficiently smooth function

†*Prikl. Mat. Mekh.* Vol. 57, No. 1, pp. 58–64, 1993.

of the flow integrals is by nature this very integral, the functions $\rho(\zeta)$, $U(\eta, \zeta)$ and the total energy $H(\eta, \zeta)$ are the integrals of the equations of motion, and the Bernoulli integral

$$p + \nu\rho(\zeta)z + \tfrac{1}{2}\rho(\zeta)U^2(\eta,\zeta)b^2 = H(\eta,\zeta), \quad \nu = gH_0/U_0^2 \tag{1.2}$$

holds, where p is the pressure.

By projecting Euler's equation on the directions $\nabla\eta$ and $\nabla\zeta$, and by using Eq. (1.2) with asymptotic equalities $y=\eta$ and $z=\zeta$ as $x\to-\infty$, after standard algebra we obtain a system of two quasi-linear second-order equations

$$\begin{aligned}
&\alpha^2(\eta,\zeta)(\mathrm{rot}[\nabla\eta,\nabla\zeta],\nabla\zeta) = \tfrac{1}{2}(|[\nabla\eta,\nabla\zeta]|^2 - 1)\partial\alpha^2(\eta,\zeta)/\partial\eta \\
&-\alpha^2(\eta,\zeta)(\mathrm{rot}[\nabla\eta,\nabla\zeta],\nabla\eta) = \nu\rho'(\zeta)(z-\zeta) + \tfrac{1}{2}(|[\nabla\eta,\nabla\zeta]|^2 - 1)\partial\alpha^2(\eta,\zeta)/\partial\zeta \\
&\alpha^2(\eta,\zeta) = \rho(\zeta)U^2(\eta,\zeta)
\end{aligned} \tag{1.3}$$

Let us now formulate the boundary conditions. We will denote the abrupt change in the arbitrary function Φ when passing through the kth interface by $[\Phi]_k$. If $p_{-\infty}$ is the pressure in the undisturbed flow, then

$$[p-p_{-\infty}]_k = [\eta]_k = [\zeta]_k = 0, \quad k=1,\ldots,n \tag{1.4}$$

$$\zeta|_{z=-H} = -H \tag{1.5}$$

For an infinitely deep fluid, condition (1.5) is replaced by the condition for $\nabla(\eta-y)$ and $\nabla(\zeta-z)$ to be bounded within the flow region. Besides, radiation conditions are imposed which result in a higher order of decrease in $\nabla(\eta-y)$ and $\nabla(\zeta-z)$ as $x\to-\infty$, compared to the order of decrease as $x\to+\infty$.

At the surface $\partial\Omega$ of the solid Ω the condition of impermeability

$$(\nabla\eta,\nabla\zeta,\mathrm{N}) = 0 \tag{1.6}$$

is satisfied, where N is the vector of the normal to $\partial\Omega$. In the vicinity of a non-singular point, the smooth surface $\partial\Omega$ may be defined by an explicit equation, e.g. $z=z(x, y)$. In this vicinity, condition (1.6) takes the form

$$\partial(\eta|_{\partial\Omega},\zeta|_{\partial\Omega})/\partial(x,y) = 0 \tag{1.7}$$

From (1.7) it is obvious that in the vicinity under consideration, at least one of the three conditions: $\eta=\text{const}$, $\zeta=\text{const}$ and $\zeta=f(n)$ is satisfied. The bounded smooth surface $\partial\Omega$ may be cut into some finite number of surfaces $\Sigma_1, \ldots, \Sigma_r$ and at least one of the following conditions

$$\eta|_{\Sigma_i} = \text{const}, \quad \zeta|_{\Sigma_i} = \text{const}, \quad \zeta|_{\Sigma_i} = f(\eta|_{\Sigma_i}) \tag{1.8}$$

holds on each of them.

A selection of surfaces Σ_i and one of the conditions of (1.8) at each surface is an essential element of the mathematical model for the flow around a solid, but it still does not ensure a unique solution. Following Zhukovskii's classical airfoil theory, the additional conditions at the boundary of the body around which the flow is considered may be associated with the nature of the set of critical points at the surface $\partial\Omega$, i.e. with the phase pattern of trajectories of fluid particles at the surface $\partial\Omega$. As experiment [1] shows, the phase pattern depends on the body shape and on the ranges of the physical parameters of the flow. In some important practical cases, the phase pattern allows of a fairly simply description [2].

We will consider well-streamlined bodies (e.g. convex bodies obtained by the revolution of a smooth curve around an axis parallel to x axis). If h is a characteristic dimension of the body, N is Vaisala–Brunt number and $\mathrm{Fr}=U_0Nh>1$, $U(y, z)=U_0$, then, as experiment [1] shows, the phase pattern is of the same nature as in the case when potential uniform flow around the body occurs. On the surface $\partial\Omega$ there are only two critical points A and B. The trajectory which intercepts point A branches at this point into a pencil of trajectories which later converge at point B. The boundary conditions on $\partial\Omega$ take the following form

$$\eta|_{\partial\Omega}=\eta_0, \quad \zeta|_{\partial\Omega}=\zeta_0 \tag{1.9}$$

and in the case of a solid of revolution, the constant $\eta_0=0$. The constant ζ_0 must be determined when solving the flow problem [3].

In the case of $\mathrm{Fr}<1$, the pattern of the phase trajectories on $\partial\Omega$ will be more complex. Let Ω be a solid of revolution, and let Γ be a section of $\partial\Omega$ by the xz plane. Experiment shows [1] that the critical points of the flow are located in two arcs Γ_1 and Γ_1 separated from the curve Γ by two planes $z=h_1$ and $z=h_2$. At $z\in[h_1, h_2]$ the phase curves are the sections of the $\partial\Omega$ surface with the $z=\mathrm{const}$ planes. At $z>h_2$ and $z<h_1$ the phase trajectories have the form of pencils which lie at $\partial\Omega$ and converge at the ends of the arcs Γ_1 and Γ_1. The following model of the boundary conditions on $\partial\Omega$ is proposed

$$\zeta=\zeta_1, \quad z<h_1; \quad \zeta=\zeta_2, \quad z>h_2; \quad \zeta=f(z), \quad h_1<z<h_2 \tag{1.10}$$

The function $f(z)$ increases by $[h_1, h_2]$ from ζ_1 to ζ_2. The constants h_1 and h_2 will be found from the requirement that the particles which lie on the trajectories, arriving at the ends of arcs Γ_1 and Γ_2, possess sufficient energy to reach the nearest apex of the body Ω. We may try to approximate the function $f(z)$ by a linear one.

The model proposed for the flow without separation is insufficient in the case when in the boundary layer on the body an intensive process of vortex formation and separation occurs. It is likely that in this case within the framework of the ideal fluid concept it will be necessary to develop the jet flow models. The experimental paper [4] presents data on the influence of diffusion processes, which occur in the inhomogeneous fluid, on the flow pattern.

2. LINEARIZATION

We put $\zeta=z-w$ and $\eta=y-u$, and assume that u and w will be treated as small first-order terms. Neglecting small terms of higher order in Eqs (1.1)–(1.4) we obtain

$$v_x=U(y,z)\left(1-\frac{\partial u}{\partial y}-\frac{\partial w}{\partial z}\right), \quad v_y=U(y,z)\frac{\partial u}{\partial x}, \quad v_z=U(y,z)\frac{\partial w}{\partial x} \tag{2.1}$$

$$\alpha^2(y,z)\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial y}\left(\alpha^2(y,z)\left(\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)\right)=0 \tag{2.2}$$

$$\alpha^2(y,z)\frac{\partial^2 w}{\partial x^2}+\frac{\partial}{\partial z}\left(\alpha^2(y,z)\left(\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)\right)-\nu\rho'(z)w=0$$

$$\left[\alpha^2(y,z)\left(\frac{\partial w}{\partial z}+\frac{\partial u}{\partial y}\right)-\nu f(z)w\right]_k=[u]_k=[w]_k=0 \tag{2.3}$$

$$w|_{z=-H}=0, \quad k=0,1,\ldots,n$$

When the velocity $U(y, z)$ does not depend on y, the system of equations (2.2) and boundary conditions (2.3) will be reduced to standard form

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial z}\left(\alpha^2(z)\frac{\partial w}{\partial z}\right)+\alpha^2(z)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)-\nu\rho'(z)w\right)-\nu\rho'(z)\frac{\partial^2 w}{\partial y^2}=0 \tag{2.4}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 w}{\partial y\partial z},\quad w|_{z=-H}=0$$

$$\left[\frac{\partial^2}{\partial x^2}\left(\alpha^2(z)\frac{\partial w}{\partial z}-\nu\rho w\right)-\nu\rho\frac{\partial^2 w}{\partial y^2}\right]_k=[w]_k=[u]_k=0,\quad k=0,\ldots,n$$

The linearized equations provide a good description of the flow at a fair distance from the source of the disturbance.

3. THE FORCE ACTING ON THE OBSTACLE

Let S be a closed surface containing the domain Ω within it, K_x the projection of the momentum on the x axis, and D the force caused by the action of the fluid on the obstacle. By virtue of the law of change of momentum

$$D=-\int_S(p\cos nx+\rho v_x v_n)dS \tag{3.1}$$

By substituting the value of the pressure from Bernoulli's equation (1.2) into (3.1) and denoting the pressure in the undisturbed flow by $p_0(z)$ we obtain

$$D=-\int_S\rho(v_x v_y dzdx+v_x v_z dxdy)+\int_S(\tfrac{1}{2}\rho(v_y^2+v_z^2-v_x^2-U^2)+ \\ +\nu\rho(\zeta)(z-\zeta)-p_0(\zeta))dydz \tag{3.2}$$

For any smooth function $\operatorname{div}(F(\eta,\zeta)\mathbf{v})=0$ and by virtue of Gauss' theorem we have

$$\int_S\rho(\zeta)U(\eta,\zeta)(v_x dydz+v_y dzdx+v_z dxdy)=0 \tag{3.3}$$

Using the equalities

$$p_0'(\zeta)=-\nu\rho(\zeta),\quad \int_S p_0(z)dydz=0,\quad R=2\int_0^1\tau\rho'(z+\tau w)d\tau$$

$$p_0(z)-p_0(\zeta)-\nu p_0'(\zeta)(z-\zeta)=-\tfrac{1}{2}\nu w^2 R(z,w)$$

and subtracting equality (3.3) from (3.2) we obtain

$$D=-\int_S\rho(\zeta)(v_x-U)(v_y dzdx+v_z dxdy)+ \\ +\tfrac{1}{2}\int_S(\rho(\zeta)(v_y^2+v_z^2-(v_x-U)^2)-\nu R(z,w)w^2)dydz \tag{3.4}$$

If the surface S is sufficiently far from Ω, we can substitute the linear approximations (2.1) into (3.4) and, by making use of the fact that $R(z,0)=\rho'(z)$, we obtain

$$D=-\int_S\alpha^2(y,z)\left(\frac{\partial u}{\partial y}+\frac{\partial w}{\partial z}\right)(u_x dzdx-w_x dxdy)+ \tag{3.5}$$

$$+ \tfrac{1}{2}\int_S (\alpha^2(y,z)\left(u_x^2 + w_x^2 - \left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)^2\right) - \nu\rho'(z)w^2)dydz$$

In the plane case, the surface integral must be replaced by a curvilinear one and we must put $u=0$ [5].

Let us transform formula (3.5) by using the system of equations and boundary conditions (2.1)–(2.3).

By multiplying the first equation (2.2) by w and the second equation by u, subtracting the first result from the second and substituting the expression obtained into formula (3.5), we find that

$$D = \frac{1}{2}\int_S \alpha^2(y,z)\left(\left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2}\right)dydz +$$

$$+ \frac{1}{2}\int_S\left(\frac{\partial}{\partial y}\left(\alpha^2(y,z)u\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(\alpha^2(y,z)w\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)\right)\right)dydz -$$

$$- \frac{1}{2}\int_S \alpha^2(y,z)\left(\frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)dzdx - \frac{\partial w}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial z}\right)dxdy\right)$$

Let us take S as a side of the parallelepiped $\xi_1 \leqslant x \leqslant \xi$, $-\eta \leqslant y \leqslant \eta$, $-H \leqslant z \leqslant 0$. Letting $\xi_1 \to -\infty$, $\xi \to +\infty$, $\eta \to +\infty$ and taking account of the boundary conditions at the bottom and on the free boundary, we obtain

$$D = \frac{1}{2}\lim_{x\to+\infty}\int_{-\infty}^{+\infty}\int_{-H}^{0}\alpha^2(y,z)\left(\left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 - w\frac{\partial^2 w}{\partial x^2}\right)dydz \qquad (3.6)$$

In the case of a fluid of infinite depth, $H=+\infty$.

4. THE UNBOUNDED FLOW OF AN EXPONENTIALLY STRATIFIED FLUID AROUND A SPHERE

We take the sphere radius R as a length unit, denote the Vaisala–Brunt frequency by N and assume that

$$\nu = \frac{gR}{U^2}, \quad \beta = \frac{NR}{U}, \quad \epsilon = \frac{N^2R}{g}, \quad \omega = e^{-\frac{1}{2}\epsilon z}w, \quad \theta = e^{-\frac{1}{2}\epsilon z}u$$

Assuming that the parameter ϵ is much less than β, we transform Eqs (2.4) to the form

$$L\omega = \frac{\partial^2}{\partial x^2}(\Delta\omega + \beta^2\omega) + \beta^2\frac{\partial^2\omega}{\partial y^2} = 0, \quad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = -\frac{\partial^2\omega}{\partial y\partial z} \qquad (4.1)$$

Assuming that ω and θ are integrable with a square in the plane $x=\text{const}$ denoting the corresponding plane Fourier transformation by $F\omega$ and $F\theta$ we obtain from (3.6) by means of Plancherel's equation

$$D = \frac{1}{8\pi^2}\lim_{x\to+\infty}\int_{-\infty}^{+\infty}\left(\left(\frac{\partial F\theta}{\partial x}\right)^2 - F\theta\frac{\partial^2 F\theta}{\partial x^2} + \left(\frac{\partial F\omega}{\partial x}\right)^2 - F\omega\frac{\partial^2 F\omega}{\partial x^2}\right)dpdq \qquad (4.2)$$

If the far field is approximated by the field of the flow past a dipole, the functions θ^2 and ω^2

are found to be non-integrable and Eq. (4.2) is unsuitable. To overcome this difficulty, the flow around a sphere will be approximated by the flow around a system of dipoles distributed over the surface of the sphere. The distribution density depends analytically on the parameter β [3] and to a first approximation we can take $\beta=0$ in the expression for the density distribution, i.e. we can take this density to be the same as in the problem of the flow of a homogeneous fluid around a sphere.

The source function $G(x, y, z)$ is a solution of the equation $LG=\delta''(x)\delta(y)\delta(z)$, the operator L being defined by (4.1), and the delta-function is denoted by δ. The usual procedure results in the following formula

$$FG = \tfrac{1}{2}Cde^{-xC} + Ad^{-1}\sin xA, \quad 2A^2 = d+\beta^2-p^2-q^2 \tag{4.3}$$
$$2C^2 = d+p^2+q^2-\beta^2, \quad d=(\beta^2-p^2-q^2)^2+4\beta^2q^2$$

which holds for $x>0$.

The exponentially decreasing term in (4.3) can be ignored, as it does not affect the value of the force D calculated from (4.2).

Let us consider the problem of the unbounded flow of a homogeneous fluid around a sphere of unit radius. Using the formula [6]

$$\psi = \tfrac{1}{2}\rho^2(1-r^{-3}), \quad r^2=x^2+y^2+z^2, \quad \rho^2=y^2+z^2$$

and the relations

$$\rho^2(1-r^{-3}) = \eta^2+\zeta^2, \quad \zeta/\eta = z/y$$

we obtain

$$w = z-\zeta = z(1-\sqrt{1-r^{-3}})$$

On the boundary of the sphere $\zeta=0$ and $w=z$, and as $r\to+\infty$ the asymptotic equality $w=\tfrac{1}{2}zr^{-3}$ holds. Note that $\tfrac{1}{2}zr^{-3}$ is a harmonic function taking the value $\tfrac{1}{2}z$ on the sphere boundary. If the linearized equation $\Delta w=0$, is used, then to obtain the proper asymptotic forms for w as $r\to\infty$ it is necessary to solve the linear equation $\Delta w=0$ with boundary condition $w|_{\partial\Omega}=\tfrac{1}{2}z$. Such a solution can be represented in the form of a double-layer potential with density $3\zeta/(8\pi)$.

Approximating the inhomogeneous fluid flow around the sphere by the corresponding double-layer potential [3] with density $3\zeta/(8\pi)$, we obtain

$$\omega = \frac{3}{2}\int_{\partial\Omega}\zeta\Big(\xi\frac{\partial}{\partial\xi}G(x-\xi, y-\eta, z-\zeta)+\eta\frac{\partial G(\cdot)}{\partial\eta}+\zeta\frac{\partial G(\cdot)}{\partial\zeta}\Big)\,dS$$

Omitting the exponential term in Eq. (4.3) for the Fourier transformation and using the properties of the Fourier transformation, convolution and the symmetry of the sphere, we obtain

$$F\omega = \frac{3}{2}\,d^{-1}\sin Ax\int_{\partial\Omega} i\zeta(A\xi+p\eta+q\zeta)e^{i(A\xi+p\eta+q\zeta)}dS \tag{4.4}$$

When the orthogonal linear transformation

$$\xi' = \gamma^{-1}(A\xi+p\eta+q\zeta), \quad \eta'=\alpha^{-1}(p\zeta-q\eta)$$
$$\zeta' = (\alpha\gamma)^{-1}(Ap\eta+Aq\zeta-\alpha^2\xi), \quad \alpha^2=p^2+q^2, \quad \gamma^2=A^2+p^2+q^2 \tag{4.5}$$

is applied, the sphere is transformed into itself, $\zeta' = q\gamma^{-1}\xi' + p\alpha^{-1}\eta' + qA(\alpha\gamma)^{-1}\zeta'$. From (4.4), using the symmetry of the sphere, we obtain

$$F\omega = 6\pi d^{-1} Aq\chi(\gamma)\sin Ax$$
$$\chi(\gamma) = 2\gamma^{-3}(\gamma\cos\gamma - \sin\gamma) - \gamma^{-1}\sin\gamma \tag{4.6}$$

Setting up a relation between Fθ and Fω, by means of (4.1) and substituting the value of Fω from (4.6) into (4.2), we obtain

$$D = \frac{9}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^{-2}q^2\chi^2(\gamma)A^4\left(1 + \frac{p^2q^2}{(A^2+q^2)^2}\right)dp\,dq \tag{4.7}$$

Using (4.3) and making the change of variables

$$p = \beta\sqrt{t}\cos\varphi,\quad q = \beta\sqrt{t}\sin\varphi,\quad d^2 = \beta^4 d_1^2,\quad d_1^2 = (1+t)^2 - 4t\cos^2\varphi$$
$$A^2 = \beta^2 A_1^2,\quad A_1^2 = \tfrac{1}{2}(d_1(t) + 1 - t),\quad \gamma(t) = \beta\gamma_1(t),\quad \gamma_1(t) = \sqrt{A_1^2 + t}$$

we can write (4.7) in the form

$$\beta^{-4}D = \frac{9}{4}\int_0^{2\pi}\int_0^{+\infty} tA_1^4\chi^2(\beta\gamma_1)d_1^{-2}\left(\sin^2\varphi + \frac{t^4\sin^4\varphi\cos^2\varphi}{(A_1^2 + t^2\sin^2\varphi)^2}\right)dt\,d\varphi \tag{4.8}$$

In Eq. (4.8) we shall find the principal term of the asymptotic form as $\beta \to 0$. Note that as $t \to +\infty$

$$d_1 \sim t,\quad A_1 \sim |\sin\varphi|,\quad \gamma_1 \sim \sqrt{t},\quad \sin^2\varphi + \frac{t^4\sin^4\varphi\cos^2\varphi}{(A_1^2 + t^2\sin^2\varphi)^2} \sim 1 \tag{4.9}$$

Let T be so large that we can use formulae (4.9) when $t > T$. If the inner integral in (4.8) is represented in the form of the sum of integrals over the intervals $(0, T)$ and $(T, +\infty)$, the first integral is bounded, and the second has the order of magnitude $\ln\beta$ as $\beta \to 0$, and hence, it defines the asymptotic form as $\beta \to 0$. In fact, by virtue of (4.9)

$$D = \frac{27}{16}\beta^4\int_{\beta T}^{+\infty}\chi^2(\tau)d\tau/\tau =$$
$$= -\frac{27}{16}\beta^4\chi^2(\beta T)\ln(\beta T) + \frac{27}{8}\beta^4\int_{\beta T}^{+\infty} 2\chi(\tau)\chi'(\tau)\ln\tau\,d\tau$$

It follows from (4.6) that $\chi^2(0) = 1/9$, and the integral

$$\int_0^{+\infty}\chi\chi'\ln\tau\,d\tau$$

converges. Therefore, as $\beta \to 0$

$$D = \frac{3\pi}{16}\beta^4|\ln\beta|,\quad \beta = \frac{NR}{U} \tag{4.10}$$

Many investigators have simulated the flow around a sphere by distributing the mass sources over its surface with a density taken from the solution of the corresponding problem for a homogeneous fluid.† In this case, the coefficient in formula (4.10) is found to be equal to $\pi/8$. The discrepancy is probably connected with the impossibility of making a choice of variables for which the solution of the problem of the flow of an inhomogeneous fluid around a body reduces to the solution of the Neumann problem as in the case of the flow of a homogeneous fluid around a body. If we seek the solution in the form of a simple-layer potential, the problems of the solvability of the corresponding integral equations and the continuous dependence of the solutions upon the parameter have not been studied.

In dimensional variables the force acting on the sphere is equal to $\rho_0(RU)^2D$. Taking, for comparison, the force of viscous resistance as given by Stokes' formula $W=6\pi\rho_0(RU)^2\,\mathrm{Re}^{-1}$, we find that the ratio of the wave drag to the viscous drag is equal to $\beta^4|\ln\beta|\,\mathrm{Re}/32$.

REFERENCES

1. SNYDER W. H., THOMSON R. S. and ESKRIDGE R. E., The structure of strongly stratified flow over hills: Dividing-streamline concept. *J. Fluid Mech.* **152**, 249–288, 1985.
2. BEZHANOV K. A., ONUFRIYEV A. T. and TER-KRIKOROV A. M., Investigation of the far and near fields in problems of exponentially stratified flow past a bottom irregularity. *Izv. Akad. Nauk SSSR. MZhG* 5, 86–94, 1987.
3. TER-KRIKOROV A. M., Stratified flows and dipole approximations. *Prikl. Mat. Mekh.* **54**, 587–592, 1990.
4. SYSOYEVA Ye. A. and CHASHECHKIN Yu. D., Vortex systems of wake stratified flow behind a sphere. *Izv. Akad. Nauk SSSR. MZhG* 4, 82–90, 1991.
5. MILES J., Lee waves in a stratified flow. Pt. 1. Thin barrier. *J. Fluid Mech.* **32**, 549–567, 1968.
6. KOCHIN N. Ye., KIBEL I. A. and ROZE N. V., *Theoretical Hydromechanics*, Vol. 1. Gostekhizdat, Moscow, 1955.

Translated by Yu.P.G.

†See the details in the unpublished paper by GORODTSOV V. A. and TEODOROVICH E. V., Cherenkov radiation of inner waves by uniformly moving sources. Preprint No. 183, Institute for Problems in Mechanics, Moscow, 1981.